

DETERMINANTS OF GRIDS, TORI, CYLINDERS AND MÖBIUS LADDERS

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ABSTRACT. Recently, Bieñ [A. Bieñ, The problem of singularity for planar grids, *Discrete Math.* 311 (2011), 921–931] obtained a recursive formula for the determinant of a grid. Also, recently, Prager [D. Prager, Determinants of box products of paths, *Discrete Math.* 312 (2012), 1844–1847], independently, obtained an explicit formula for this determinant. In this paper, we give a short proof for this problem. Furthermore, applying the same technique, we get explicit formulas for the determinant of a torus, a cylinder, and a Möbius ladder.

1. INTRODUCTION AND RESULTS

We denote by $A(G)$ the adjacency matrix of a graph G . A path (cycle) on n vertices is denoted by P_n (resp., C_n). Given two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, their *Cartesian product* $G_1 \square G_2$ is the graph with vertex set $V_1 \times V_2$ and edge set

$$\left\{((u, v), (u', v)) : (u, u') \in E_1, v \in V_2\right\} \cup \left\{((u, v), (u, v')) : u \in V_1, (v, v') \in E_2\right\}.$$

Cartesian product produces many important classes of graphs. For example, a *grid* (also called *mesh*) is the Cartesian product of two paths, a *torus* (also called *toroidal grid* or *toroidal mesh*) is the Cartesian product of two cycles, and a *cylinder* is the Cartesian product of a path and a cycle. One can generalize these definitions to more than two paths or cycles. These classes of graphs are widely used computer architectures (e.g., grids are widely used in multiprocessor VLSI systems) [7].

The *nullity* of a graph G of order n , denoted by $\eta(G)$, is the multiplicity of 0 in the spectrum of G . Clearly, $\eta(G) = n - r(A(G))$, where $r(A(G))$ is the rank of $A(G)$. The nullity of a graph is closely related to the minimum rank problem of a family of matrices associated with a graph (see, e.g., [5] and the references therein). Nullity of a (molecular) graph (specifically, determining whether it is positive or zero) has also important applications in quantum chemistry and Hückel molecular orbital (HMO) theory (see, e.g., [6] and the references therein). A famous problem, posed by Collatz and Sinogowitz in 1957 [4], asks to characterize all graphs with positive nullity. Clearly, $\det A(G) = 0$ if and only if $\eta(G) > 0$. So, examining the determinant of a graph is a way to attack this problem. But there seems to be little published on calculating the determinant of various classes of graphs.

Recently, Bieñ [2] obtained a recursive formula for the determinant of a grid. Also, recently, Prager [8], independently, obtained an explicit formula for this determinant (see (1) below). Here, using trigonometric identities, we give a short proof for this problem. Furthermore, applying the same technique, we get explicit formulas for the determinant of a torus, and a cylinder.

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Theorem 1. *Let $m > 1$ and $n > 1$ be integers. Then*

$$\det A(P_{m-1} \square P_{n-1}) = \begin{cases} (-1)^{\frac{(m-1)(n-1)}{2}} & \text{if } \gcd(m, n) = 1; \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

$$\det A(C_m \square C_n) = \begin{cases} 4^{\gcd(m, n)} & \text{if } m \text{ and } n \text{ are odd;} \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

$$\det A(P_{m-1} \square C_n) = \begin{cases} m & \text{if } n \text{ is odd and } \gcd(m, n) = 1; \\ (-1)^{m-1} m^2 & \text{if } n \text{ is even and } \gcd(m, n/2) = 1; \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Note that for $m = 2$, (1) and (3) give the following well-known determinants.

$$\det A(P_{n-1}) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad \det A(C_n) = \begin{cases} 2 & \text{if } n \text{ is odd;} \\ -4 & \text{if } n \equiv 2 \pmod{4}; \\ 0 & \text{otherwise.} \end{cases}$$

Our proof techniques or its modifications may be useful in other situations with similar flavor (see, e.g., [1]). For example, let us consider the *Möbius ladder* M_{2n} , the graph on $2n$ vertices whose edge set is the union of the edge set of C_{2n} and $\{(v_i, v_{n+i}) : i = 1, \dots, n\}$. We prove that

Theorem 2. *Let $n > 1$ be an integer. Then*

$$\det A(M_{2n}) = \begin{cases} -3 & \text{if } n \equiv \pm 2 \pmod{6}; \\ -9 & \text{if } n \equiv \pm 1 \pmod{6}; \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

2. TECHNIQUES AND PROOFS

The starting point of our calculations is the following well-known theorem which gives the eigenvalues of the Cartesian product of two graphs (see, e.g., [9, p. 587]).

Theorem 3. *Let G_1 be a graph of order m , and G_2 be a graph of order n . If the eigenvalues of $A(G_1)$ and $A(G_2)$ are, respectively, $\lambda_1(G_1), \dots, \lambda_m(G_1)$ and $\lambda_1(G_2), \dots, \lambda_n(G_2)$, then the eigenvalues of $A(G_1 \square G_2)$ are precisely the numbers $\lambda_i(G_1) + \lambda_j(G_2)$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.*

We also need the following trigonometric identities, which might be of independent interest.

Lemma 4. *Let n be a positive integer and let $a \in \mathbb{Z}$ such that $\gcd(a, n) = 1$. Then for any real number x ,*

$$\sin(nx) = 2^{n-1} (-1)^{\frac{(a-1)(n-1)}{2}} \prod_{j=0}^{n-1} \sin\left(x + \frac{aj\pi}{n}\right). \quad (5)$$

Moreover,

$$\prod_{j=1}^{n-1} \sin\left(\frac{aj\pi}{n}\right) = (-1)^{\frac{(a-1)(n-1)}{2}} \cdot \frac{n}{2^{n-1}} \quad (6)$$

and

$$\prod_{j=1}^{n-1} \cos\left(\frac{aj\pi}{n}\right) = \begin{cases} (-1)^{\frac{a(n-1)}{2}} \cdot \frac{1}{2^{n-1}} & \text{if } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Proof. Let $\omega = e^{\pi a I / n}$, where $I = \sqrt{-1}$. Then, since $\{\omega^{-2j} : j = 0, \dots, n-1\}$ are all the n -th roots of unity, it follows that

$$\prod_{j=0}^{n-1} (z - \omega^{-2j}) = z^n - 1.$$

Hence,

$$\begin{aligned}
\prod_{j=0}^{n-1} \sin \left(x + \frac{aj\pi}{n} \right) &= \prod_{j=0}^{n-1} \frac{e^{Ix} \omega^j - e^{-Ix} \omega^{-j}}{2I} \\
&= \frac{e^{-nIx} \omega^{n(n-1)/2}}{(2I)^n} \prod_{j=0}^{n-1} (e^{2Ix} - \omega^{-2j}) \\
&= \frac{I^{(a-1)(n-1)}}{2^{n-1}} \cdot \frac{e^{nIx} - e^{-nIx}}{2I} \\
&= \frac{(-1)^{\frac{(a-1)(n-1)}{2}}}{2^{n-1}} \cdot \sin(nx).
\end{aligned}$$

Moreover,

$$\prod_{j=1}^{n-1} \sin \left(\frac{aj\pi}{n} \right) = \frac{(-1)^{\frac{(a-1)(n-1)}{2}}}{2^{n-1}} \lim_{x \rightarrow 0} \frac{\sin(nx)}{\sin x} = (-1)^{\frac{(a-1)(n-1)}{2}} \cdot \frac{n}{2^{n-1}}.$$

Finally,

$$\prod_{j=1}^{n-1} \cos \left(\frac{aj\pi}{n} \right) = (-1)^{n-1} \prod_{j=1}^{n-1} \sin \left(-\frac{\pi}{2} + \frac{aj\pi}{n} \right) = (-1)^{n-1} \sin(n\pi/2) \cdot \frac{(-1)^{\frac{(a-1)(n-1)}{2}}}{2^{n-1}},$$

which easily yields the required formula. \square

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. It is known (see, e.g., [9, p. 588]) that the eigenvalues of $A(P_{m-1})$ and $A(C_n)$ are, respectively,

$$\left\{ 2 \cos \left(\frac{i\pi}{m} \right) : 1 \leq i \leq m-1 \right\} \quad \text{and} \quad \left\{ 2 \cos \left(\frac{2j\pi}{n} \right) : 1 \leq j \leq n \right\}.$$

The proof is done by a direct combination of Theorem 3 and Lemma 4.

We start with (1). Using the identity $\cos(a+b) + \cos(a-b) = 2 \cos(a) \cos(b)$,

$$\begin{aligned}
\det A(P_{m-1} \square P_{n-1}) &= \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \left(2 \cos \left(\frac{i\pi}{m} \right) + 2 \cos \left(\frac{j\pi}{n} \right) \right) \\
&= 2^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} 2 \cos \left(\frac{i\pi}{2m} + \frac{j\pi}{2n} \right) \cos \left(\frac{i\pi}{2m} - \frac{j\pi}{2n} \right) \\
&= 2^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} 2 \cos \left(\frac{i\pi}{2m} + \frac{j\pi}{2n} \right) \cos \left(\frac{i\pi}{2m} - \frac{(n-j)\pi}{2n} \right) \\
&= 2^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} 2 \cos \left(\frac{i\pi}{2m} + \frac{j\pi}{2n} \right) \sin \left(\frac{i\pi}{2m} + \frac{j\pi}{2n} \right) \\
&= 2^{(m-1)(n-1)} \prod_{i=1}^{m-1} \prod_{j=1}^{n-1} \sin \left(\frac{i\pi}{m} + \frac{j\pi}{n} \right) \\
&= 2^{(m-1)(n-1)} \prod_{i=1}^{m-1} \frac{\sin \left(\frac{ni\pi}{m} \right)}{2^{n-1} \sin \left(\frac{i\pi}{m} \right)} = \frac{\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{m} \right)}{\prod_{i=1}^{m-1} \sin \left(\frac{i\pi}{m} \right)},
\end{aligned}$$

where in the last but one step we have used the identity (5). Clearly, if $\gcd(m, n) \neq 1$ then $\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{m} \right) = 0$, otherwise we use (6).

Now, we show (2). In the case that m or n is even the proof is straightforward because one of the eigenvalues of $A(C_m \square C_n)$ is zero. Assume that m and n are odd and let $d = \gcd(m, n)$,

with $m' = m/d$, $n' = n/d$.

$$\begin{aligned}
\det A(C_m \square C_n) &= \prod_{i=1}^m \prod_{j=1}^n \left(2 \cos \left(\frac{2i\pi}{m} \right) + 2 \cos \left(\frac{2j\pi}{n} \right) \right) \\
&= 4^{mn} \prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \cos \left(\frac{i\pi}{m} + \frac{j\pi}{n} \right) \cos \left(\frac{i\pi}{m} - \frac{j\pi}{n} \right) \\
&= 4^{mn} \left(\prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \cos \left(\frac{i\pi}{m} + \frac{j\pi}{n} \right) \cos \left(\frac{i\pi}{m} - \frac{(n-j)\pi}{n} \right) \right) \\
&= 4^{mn} \left(\prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \cos \left(\frac{i\pi}{m} + \frac{j\pi}{n} \right) \right)^2 \\
&= 4^{mn} \left(\prod_{i=0}^{m-1} \prod_{j=0}^{n-1} \sin \left(-\frac{\pi}{2} + \frac{i\pi}{m} + \frac{j\pi}{n} \right) \right)^2 \\
&= 4^m \left(\prod_{i=0}^{m-1} \sin \left(n \left(-\frac{\pi}{2} + \frac{i\pi}{m} \right) \right) \right)^2 \\
&= 4^m \left(\prod_{i=0}^{m-1} \cos \left(\frac{ni\pi}{m} \right) \right)^2 = 4^m \left(\prod_{i=1}^{m'd} \cos \left(\frac{n'i\pi}{m'} \right) \right)^2 = 4^m \left(\frac{1}{4^{m'-1}} \right)^d = 4^d,
\end{aligned}$$

where in the last but one step we have used (7).

Finally, we prove (3).

$$\begin{aligned}
\det A(P_{m-1} \square C_n) &= \prod_{i=1}^{m-1} \prod_{j=1}^n \left(2 \cos \left(\frac{i\pi}{m} \right) + 2 \cos \left(\frac{2j\pi}{n} \right) \right) \\
&= 4^{(m-1)n} \prod_{i=1}^{m-1} \prod_{j=0}^{n-1} \cos \left(\frac{i\pi}{2m} + \frac{j\pi}{n} \right) \cos \left(\frac{i\pi}{2m} - \frac{j\pi}{n} \right) \\
&= 4^{(m-1)n} \prod_{i=1}^{m-1} \prod_{j=0}^{n-1} \cos \left(\frac{(m-i)\pi}{2m} + \frac{j\pi}{n} \right) \cos \left(\frac{(m-i)\pi}{2m} - \frac{j\pi}{n} \right) \\
&= (-4)^{(m-1)n} \prod_{i=1}^{m-1} \prod_{j=0}^{n-1} \sin \left(-\frac{i\pi}{2m} + \frac{j\pi}{n} \right) \sin \left(\frac{i\pi}{2m} + \frac{j\pi}{n} \right) \\
&= (-4)^{(m-1)n} \prod_{i=1}^{m-1} \frac{1}{4^{n-1}} \sin \left(-\frac{ni\pi}{2m} \right) \sin \left(\frac{ni\pi}{2m} \right) \\
&= (-1)^{(m-1)(n-1)} 4^{m-1} \left(\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{2m} \right) \right)^2.
\end{aligned}$$

If n is even and $\gcd(m, n') = 1$ where $n' = n/2$ then, by (6),

$$(-1)^{(m-1)(n-1)} 4^{m-1} \left(\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{2m} \right) \right)^2 = (-1)^{(m-1)} 4^{m-1} \left(\prod_{i=1}^{m-1} \sin \left(\frac{n'i\pi}{m} \right) \right)^2 = (-1)^{(m-1)} m^2.$$

If n is odd and $\gcd(m, n) = 1$ then, $\gcd(2m, n) = 1$ and by (6),

$$\begin{aligned}
(-1)^{(m-1)(n-1)} 4^{m-1} \left(\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{2m} \right) \right)^2 &= 4^{m-1} \left(\prod_{i=1}^{m-1} \sin \left(\frac{ni\pi}{2m} \right) \right) \left(\prod_{i=m+1}^{2m-1} \sin \left(\frac{n(2m-i)\pi}{2m} \right) \right) \\
&= 4^{m-1} \sin \left(\frac{n\pi}{2} \right) \prod_{i=1}^{2m-1} \sin \left(\frac{ni\pi}{2m} \right) = m
\end{aligned}$$

It is easy to verify that the remaining cases yield zero. \square

Now, we prove Theorem 2.

Proof of Theorem 2. The eigenvalues of $A(M_{2n})$ are (see, e.g., [3, p. 21])

$$\left\{ (-1)^j + 2 \cos \left(\frac{j\pi}{n} \right) : 1 \leq j \leq 2n \right\}.$$

Hence,

$$\begin{aligned} \det A(M_{2n}) &= \prod_{j=1}^{2n} \left((-1)^j + 2 \cos \left(\frac{j\pi}{n} \right) \right) \\ &= \prod_{j=0}^{2n-1} \left(2 \cos \left(\frac{(3j+1)\pi}{3} \right) + 2 \cos \left(\frac{j\pi}{n} \right) \right) \\ &= 4^{2n} \prod_{j=0}^{2n-1} \left(\cos \left(\frac{(3j+1)\pi}{6} + \frac{j\pi}{2n} \right) \cos \left(\frac{(3j+1)\pi}{6} - \frac{j\pi}{2n} \right) \right) \\ &= 4^{2n} \prod_{j=0}^{2n-1} \sin \left(\frac{\pi}{3} - \frac{(n+1)j\pi}{2n} \right) \prod_{j=0}^{2n-1} \sin \left(\frac{\pi}{3} - \frac{(n-1)j\pi}{2n} \right). \end{aligned}$$

If n is even then $\gcd(n+1, 2n) = 1$, and then by (5),

$$\det A(M_{2n}) = -4 \sin^2 \left(\frac{2n\pi}{3} \right) = \begin{cases} -3 & \text{if } n \equiv \pm 2 \pmod{6}; \\ 0 & \text{if } n \equiv 0 \pmod{6}. \end{cases}$$

If n is odd then $\gcd(n'+1, n) = 1$, where $n' = (n-1)/2$, and then by (5),

$$\begin{aligned} \det A(M_{2n}) &= -4^{2n} \left(\prod_{j=0}^{n-1} \sin \left(\frac{\pi}{3} - \frac{(n'+1)j\pi}{n} \right) \right)^4 \\ &= -16 \sin^4 \left(\frac{n\pi}{3} \right) = \begin{cases} -9 & \text{if } n \equiv \pm 1 \pmod{6}; \\ 0 & \text{if } n \equiv 3 \pmod{6}. \end{cases} \end{aligned}$$

\square

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